

Sign Change Methods

Example

Show that $f(x) = x^3 - 3x + 1$ has a root between $x = 1$ and $x = 2$.

$$f(1) = 1 - 3 + 1 = -1 < 0$$

$$f(2) = 8 - 6 + 1 = 3 > 0$$

Since f is continuous and there is a sign change on $[1, 2]$, there is a root in $(1, 2)$.

Example

How can a sign change method fail?

- If the curve touches the x -axis (a repeated root), there may be no sign change.
- If f has a vertical asymptote in the interval, the sign may change without a root existing.

Tip

When you have found an approximate root, you must *verify* it to the required accuracy using a sign change. For example, to show that $\alpha = 1.35$ to 2 decimal places, check that $f(1.345)$ and $f(1.355)$ have opposite signs.

Fact (Intermediate Value Theorem) — If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then f has at least one root in (a, b) .

Bisection Method

Fact (Bisection) — Given an interval $[a, b]$ with $f(a)$ and $f(b)$ of opposite sign:

1. Find the midpoint $m = \frac{a+b}{2}$.
2. Evaluate $f(m)$.
3. If $f(a)$ and $f(m)$ have opposite signs, the root is in $[a, m]$. Otherwise it is in $[m, b]$.
4. Repeat with the new interval.

Each step halves the width of the interval, so after n steps the root is known to within $\frac{b-a}{2^n}$.

Example

Use the bisection method to find the root of $f(x) = x^3 - 3x + 1$ in $[1, 2]$, correct to 1 decimal place.

Step	a	b	$m = \frac{a+b}{2}$	$f(m)$
1	1	2	1.5	$1.5^3 - 3(1.5) + 1 = -0.125$
2	1.5	2	1.75	$1.75^3 - 3(1.75) + 1 = 1.109\dots$
3	1.5	1.75	1.625	$1.625^3 - 3(1.625) + 1 = 0.416\dots$
4	1.5	1.625	1.5625	$1.5625^3 - 3(1.5625) + 1 = 0.127\dots$
5	1.5	1.5625	1.53125	< 0

The root lies in $[1.5, 1.5625]$.

Check: $f(1.55) = 0.274\dots > 0$ and $f(1.45) = -0.299\dots < 0$, so the root is 1.5 to 1 d.p.

Decimal Search

Fact (Decimal Search) — Starting from an interval $[a, b]$ containing a root:

1. Divide $[a, b]$ into 10 equal sub-intervals.
2. Evaluate f at each division point to find the sub-interval where the sign changes.
3. Repeat with the narrower interval.

Each stage gives one more decimal place of accuracy.

Example

Use decimal search to find the root of $f(x) = x^3 - 3x + 1$ in $[1, 2]$ correct to 2 decimal places.

Stage 1: Evaluate at $x = 1.0, 1.1, \dots, 2.0$:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$f(x)$	-1	-0.97	-0.87	-0.70	-0.46	-0.13	+0.30	+0.81	+1.43	+2.16

Sign change between 1.5 and 1.6, so root $\in [1.5, 1.6]$.

Stage 2: Evaluate at $x = 1.50, 1.51, \dots, 1.60$:

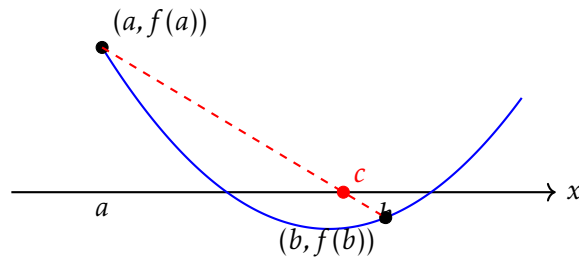
x	1.50	1.51	1.52	1.53	1.54
$f(x)$	-0.125	-0.090	-0.055	-0.019	+0.018

Sign change between 1.53 and 1.54, so root $\in [1.53, 1.54]$.

Check: $f(1.535) = -0.0004\dots < 0$ and $f(1.545) = 0.037\dots > 0$, but both 1.535 and 1.534 round to 1.53 to 2 d.p., so the root is 1.53 to 2 d.p.

Linear Interpolation

Instead of taking the midpoint (bisection), we can draw the straight line between $(a, f(a))$ and $(b, f(b))$ and use its x -intercept as our next approximation:



Example

Derive a formula for c , the x -intercept of the line through $(a, f(a))$ and $(b, f(b))$.

The line through $(a, f(a))$ and $(b, f(b))$ has gradient $m = \frac{f(b) - f(a)}{b - a}$ and passes through $(a, f(a))$:

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

Setting $y = 0$:

$$-f(a) = \frac{f(b) - f(a)}{b - a}(c - a)$$

$$c - a = \frac{-f(a)(b - a)}{f(b) - f(a)}$$

$$c = a - f(a) \cdot \frac{b - a}{f(b) - f(a)} = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Example

Use linear interpolation to find a better approximation to the root of $f(x) = x^3 - 3x + 1$ in $[1, 2]$.

$$f(1) = -1, f(2) = 3.$$

$$\begin{aligned}c &= \frac{1 \cdot f(2) - 2 \cdot f(1)}{f(2) - f(1)} \\ &= \frac{1 \cdot 3 - 2 \cdot (-1)}{3 - (-1)} \\ &= \frac{5}{4} = 1.25\end{aligned}$$

$$f(1.25) = 1.25^3 - 3(1.25) + 1 = -0.796... < 0$$

So the root is in $[1.25, 2]$. We can repeat:

$$c = \frac{1.25 \cdot 3 - 2 \cdot (-0.796...)}{3 - (-0.796...)} = \frac{5.343...}{3.796...} = 1.407...$$

After just 2 steps we are already close to the true root ≈ 1.532 .

Tip

Linear interpolation usually converges faster than bisection, but it is not guaranteed to. Bisection is slower but *always* halves the interval.

Iterative Methods

Fact (Fixed-Point Iteration) — To solve $f(x) = 0$, rearrange to the form $x = g(x)$. Then the iteration

$$x_{n+1} = g(x_n)$$

may converge to a root, provided $|g'(\alpha)| < 1$ near the root α .

Example

We want to solve $x^3 - 3x + 1 = 0$ near $x = 1.5$. Consider two rearrangements:

(a) $x = \frac{x^3 + 1}{3}$

(b) $x = \sqrt[3]{3x - 1}$

Starting from $x_0 = 1.5$, find the first 5 iterates of each. Which converges?

(a) $g(x) = \frac{x^3 + 1}{3}$

n	x_n
0	1.5
1	1.45833
2	1.36716
3	1.18514
4	0.88820
5	0.56690

This diverges away from the root near 1.5 — the iterates decrease monotonically.

(In fact $g'(x) = x^2$, and $g'(1.5) = 2.25 > 1$.)

(b) $g(x) = \sqrt[3]{3x - 1}$

n	x_n
0	1.5
1	1.51829
2	1.52619
3	1.52957
4	1.53102
5	1.53163

This converges. (Here $g'(x) = (3x - 1)^{-2/3}$ and $g'(1.53) \approx 0.39 < 1$.)

Note: *this converges to the root near 1.532.*

Tip

The same equation can be rearranged in many ways — some converge and some don't. If one rearrangement diverges, try another!

Example

The iteration $x_{n+1} = \sqrt[3]{3x_n - 1}$ appears to converge to 1.5320... Show that the root is 1.532 correct to 3 decimal places.

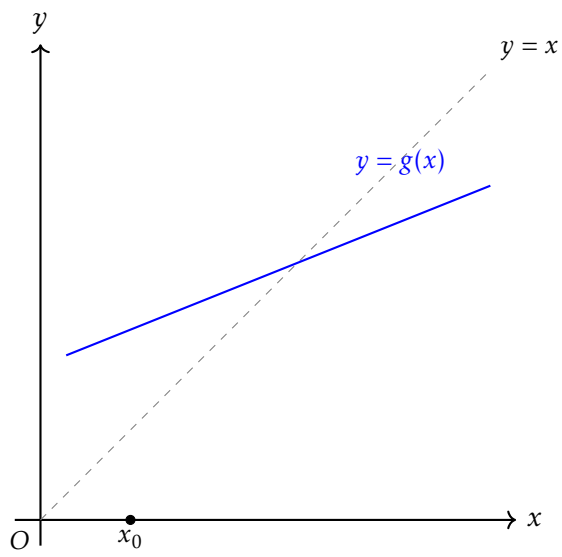
We need to show a sign change of $f(x) = x^3 - 3x + 1$ on an interval that rounds to 1.532:

$$f(1.53205) = 1.53205^3 - 3(1.53205) + 1 = -0.000157... < 0$$

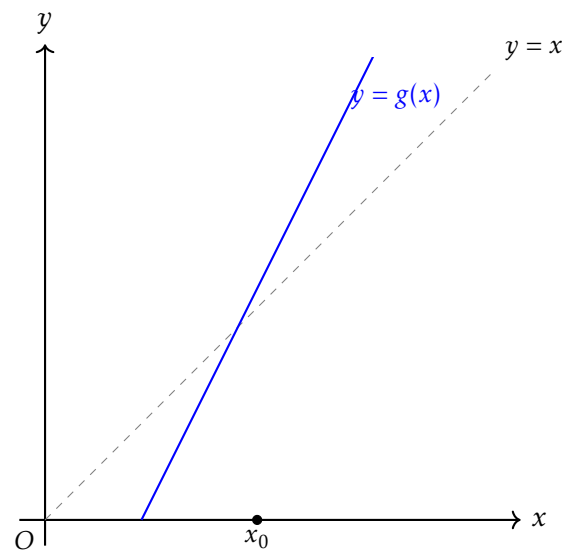
$$f(1.53215) = 1.53215^3 - 3(1.53215) + 1 = 0.000247... > 0$$

Sign change on $[1.53205, 1.53215]$, and every value in this interval rounds to 1.532 to 3 d.p. So $\alpha = 1.532$ to 3 d.p. \square

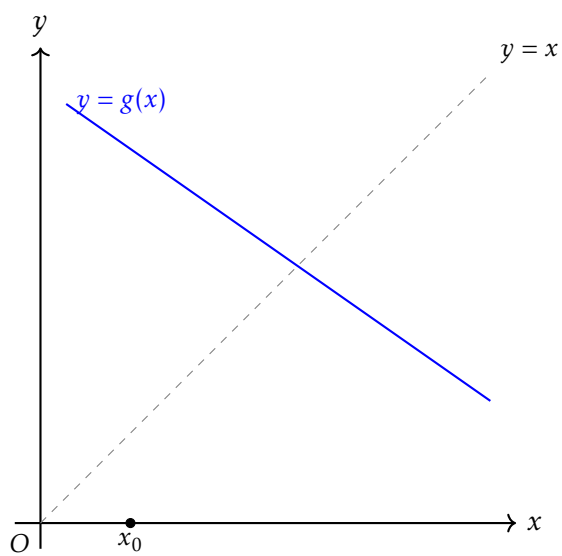
Staircase and Cobweb Diagrams



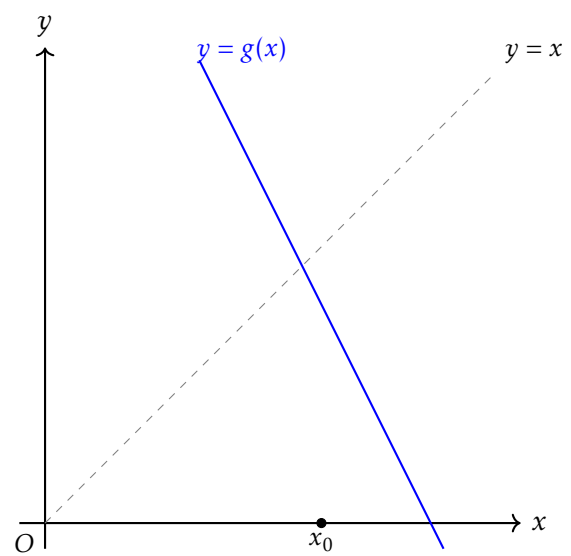
(a)



(b)



(c)



(d)

Example (OCR FP1 June 2014 Q4, adapted)

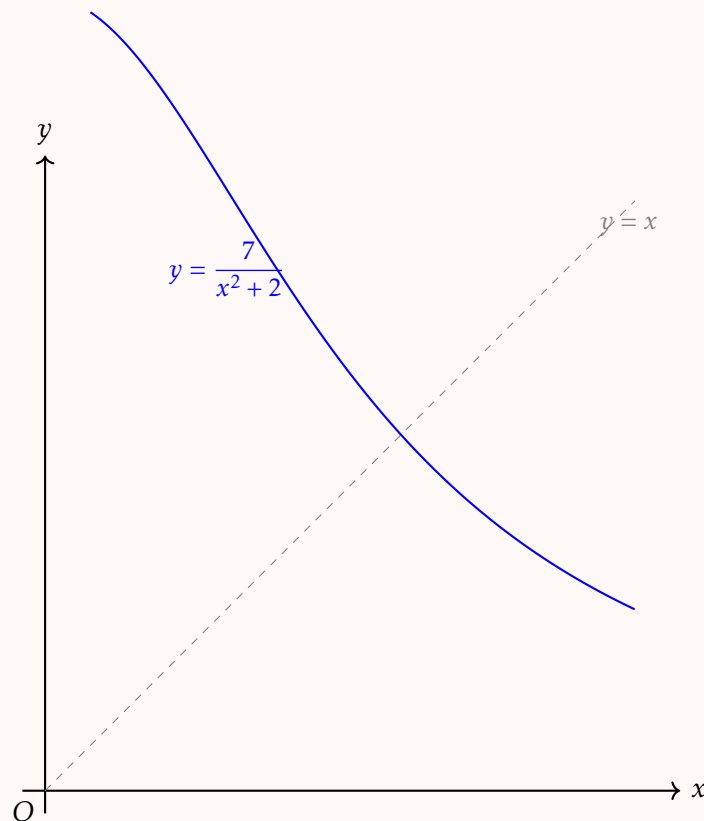
The equation $x^3 + 2x - 7 = 0$ has a root α in the interval $[1, 2]$.

(i) Show that the equation can be rearranged to give $x = \frac{7}{x^2 + 2}$ and explain why the iteration

$$x_{n+1} = \frac{7}{x_n^2 + 2} \text{ converges to } \alpha. \quad [3]$$

(ii) Starting from $x_0 = 1.5$, find x_1, x_2, x_3, x_4 , giving your answers to 4 d.p. [2]

(iii) On a copy of the diagram below, draw a cobweb or staircase diagram to illustrate this iteration. [3]



(i) $x^3 + 2x = 7 \Rightarrow x(x^2 + 2) = 7 \Rightarrow x = \frac{7}{x^2 + 2}$.

Here $g(x) = \frac{7}{x^2 + 2}$, so $g'(x) = \frac{-14x}{(x^2 + 2)^2}$.

At $\alpha \approx 1.4$: $g'(1.4) = \frac{-14 \times 1.4}{(1.96 + 2)^2} = \frac{-19.6}{15.68} \approx -0.61$.

Since $|g'(\alpha)| \approx 0.61 < 1$, the iteration converges. (And since $g'(\alpha) < 0$, we get a cobweb pattern.)

n	x_n
0	1.5
1	1.6471
2	1.4788
3	1.5622
4	1.5168

(ii)

The iterates alternate above and below α (the cobweb pattern), converging to $\alpha \approx 1.5378$.

(iii) Starting from $x_0 = 1.5$: go up to the curve, across to $y = x$ (giving $x_1 = 1.647$), up/down to the curve, across to $y = x$ (giving $x_2 = 1.479$), and so on — the iterates spiral inwards towards the intersection.

Fact — There are four cases, determined by the gradient of $y = g(x)$ at the fixed point:

Gradient $g'(\alpha)$	Pattern	Behaviour
$0 < g'(\alpha) < 1$	Staircase	Converges
$g'(\alpha) > 1$	Staircase	Diverges
$-1 < g'(\alpha) < 0$	Cobweb	Converges
$g'(\alpha) < -1$	Cobweb	Diverges

Speed of Convergence

The error at step n is $e_n = x_n - \alpha$, where α is the true root.

Example

Show that for the iteration $x_{n+1} = g(x_n)$, the errors satisfy $e_{n+1} \approx g'(\alpha) \cdot e_n$.

Since $g(\alpha) = \alpha$ (it's a fixed point):

$$\begin{aligned} e_{n+1} &= x_{n+1} - \alpha = g(x_n) - g(\alpha) \\ &\approx g'(\alpha)(x_n - \alpha) \quad (\text{linearising near } \alpha) \\ &= g'(\alpha) \cdot e_n \end{aligned}$$

So the errors form an approximately geometric sequence with common ratio $g'(\alpha)$.

This tells us:

- If $|g'(\alpha)| < 1$: errors shrink \Rightarrow convergence. The smaller $|g'(\alpha)|$, the faster.
- If $|g'(\alpha)| > 1$: errors grow \Rightarrow divergence.
- If $|g'(\alpha)| = 0$: something better happens...

Example

What happens if $g'(\alpha) = 0$? Show that the errors then satisfy $e_{n+1} \approx \frac{1}{2}g''(\alpha) \cdot e_n^2$.

If $g'(\alpha) = 0$, the linear approximation vanishes and we need the next term of the Taylor expansion:

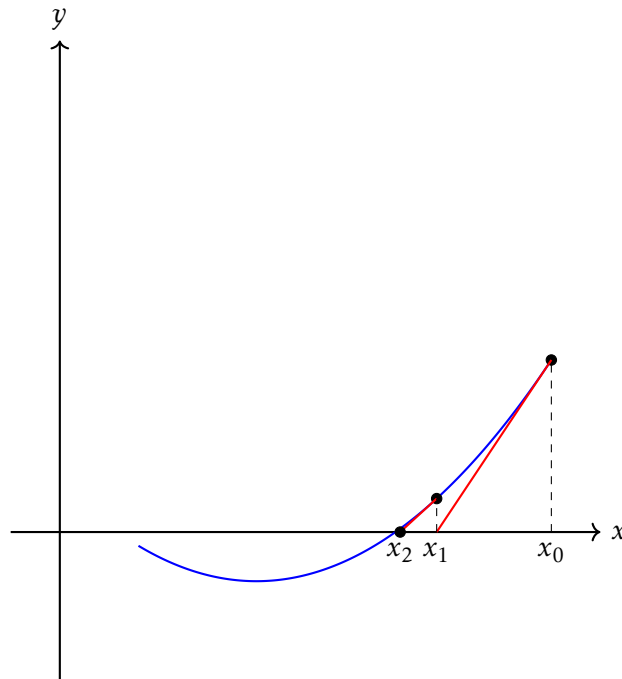
$$\begin{aligned} g(x_n) &= g(\alpha) + g'(\alpha)(x_n - \alpha) + \frac{1}{2}g''(\alpha)(x_n - \alpha)^2 + \dots \\ &= \alpha + 0 + \frac{1}{2}g''(\alpha)e_n^2 + \dots \end{aligned}$$

So $e_{n+1} = g(x_n) - \alpha \approx \frac{1}{2}g''(\alpha)e_n^2$.

This is **quadratic convergence**: each step roughly squares the error, ie the number of correct decimal places approximately doubles with each iteration.

Newton–Raphson Method

We draw the tangent to $y = f(x)$ at our current estimate x_n and take the x -intercept as x_{n+1} :



Example

Derive the Newton–Raphson formula: starting from the tangent at $(x_n, f(x_n))$, find x_{n+1} .

The tangent at $(x_n, f(x_n))$ has gradient $f'(x_n)$:

$$y - f(x_n) = f'(x_n)(x - x_n)$$

Setting $y = 0$:

$$-f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example

Newton–Raphson is the iteration $x_{n+1} = g(x_n)$ where $g(x) = x - \frac{f(x)}{f'(x)}$. Show that $g'(\alpha) = 0$ at any simple root α , and deduce that Newton–Raphson has quadratic convergence.

$$\begin{aligned}g'(x) &= 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} && \text{(quotient rule)} \\ &= \frac{f(x)f''(x)}{f'(x)^2}\end{aligned}$$

At the root, $f(\alpha) = 0$, so $g'(\alpha) = 0$.

By the result above, this means $e_{n+1} \approx \frac{1}{2}g''(\alpha)e_n^2$ — quadratic convergence. This is why Newton–Raphson roughly doubles the number of correct decimal places each step.

Example

Use the Newton–Raphson method to find the root of $f(x) = x^3 - 3x + 1$ near $x = 1.5$, correct to 4 decimal places.

$$f'(x) = 3x^2 - 3, \text{ so the iteration is } x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}.$$

n	x_n
0	1.5
1	1.533333
2	1.532092
3	1.532089

Convergence is very fast. Check: $f(1.53205) = -0.0003... < 0$, $f(1.53215) = 0.0004... > 0$, so the root is 1.5321 to 4 d.p.

Example

Use the Newton–Raphson method to find $\sqrt{5}$ to 4 decimal places.

We want to solve $f(x) = x^2 - 5 = 0$, so $f'(x) = 2x$ and

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{x_n}{2} + \frac{5}{2x_n}$$

Starting from $x_0 = 2$:

n	x_n
0	2
1	2.25
2	2.236111
3	2.236068

So $\sqrt{5} \approx 2.2361$ to 4 d.p.

Example

When can Newton–Raphson fail?

- If $f'(x_n) = 0$ at any step (a stationary point), the tangent is horizontal and never meets the x -axis.
- If the starting value is too far from the root, the iterates may diverge or cycle.

Comparison of Methods

	Method	Pros	Cons
Fact (Summary) —	Bisection	Always works (if sign change exists); guaranteed to narrow interval	Slow (only one binary digit per step)
	Decimal search	Easy to implement; systematic	Slow; lots of function evaluations
	Linear interpolation	Faster than bisection	Can be slow if the curve is very flat on one side
	Fixed-point iteration	Simple; can be fast	Rearrangement must satisfy $ g'(a) < 1$; may diverge
	Newton–Raphson	Very fast convergence	Needs $f'(x)$; can fail at stationary points

Tip

Whichever method you use, always finish by *verifying* the root to the required accuracy with a sign change!